Arithmetic: Leading the Mind toward Truth

By Thomas I. Treloar

"We must endeavor that those who are to be the principal men of our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only; … arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible and tangible objects into the argument." Plato, *The Republic*, Book VII

In the study of mathematics, the elementary school curriculum is of vital importance as it is necessary to first lay the proper foundation for further study. If the purpose of mathematics is to describe order in the universe and to hone reasoning skills, then elementary mathematics must prepare the student for such studies. Just as the typical first grader is not ready to read Shakespeare, the typical first grader is not ready to tackle the first book of Euclid's *Elements* or the quadratic formula.

To develop the critical thinking necessary for the study of advanced subjects, basic skills must be mastered and the child's mind prepared for this endeavor to be successful. In mathematics, this begins in the elementary grades with the study of arithmetic.

Arithmetic is the study of quantity and the relationships between quantities beginning with the decimal system for representing whole quantities, progressing through the elementary operations of addition, subtraction, multiplication, and division, and extending into the study of parts of a whole or fractions.

Arithmetic, at its most basic level, is an abstraction of the physical world. In some ways, it is one of the first formal abstractions to which a child is exposed, and its study is expected of every student. The meaning of a quantity and how it is represented is linked to reality, as are the meanings of the arithmetic operations. Certain situations lead to the addition of numbers, while others lead to subtraction. Still others lead to multiplication or division.

There are three meanings which can be attributed to an addition problem. They are increasing, combining, and adding the difference. If we do not know them and the subtle difference each meaning represents, we will not understand the complete picture of addition or why students almost universally struggle with one in particular, adding the difference.

To go further, each of these meanings corresponds to a meaning in subtraction. Beginning to understanding analogues and differences at this level will help students as they progress in the subject.

It is often forgotten that there is not one, but two physical meanings describing division. It is a lack of the second concrete model for division that leads to many of the difficulties in assigning meaning when the time comes to study division which involves fractions. Without this understanding we are left to rely solely on rote memorization of an algorithm for our knowledge of division by fractions.

The connection between the concrete and the abstract is one of the aspects of mathematics that had the ancient Greeks singing its praises. Mathematics is grounded on the concrete level, and this grounding is vital to initial understanding. However, the concrete can be discarded and knowledge advanced through reasoning. We should not forget, though, that we can (and occasionally must) return from the abstract to the concrete to 'test' our reasoning and understanding. This is how we can prepare a child's mind for higher thought. The connection between the concrete and the abstract displays an undeniable truth and beauty in mathematics for those who seek it.

How is the study of arithmetic to be approached? Consider the following quote from the 1960s book *A Parent's Guide to the New Math* by Evilyn Sharp.

"If you read in your child's book that 3+4=2, don't be startled. It does in the modulo five system. Also 4+1=0, 3+3=1, and 4+4=3. In other words, nothing is sacred anymore."

Contrast this quote with the words of Johannes Kepler, seventeenth century German physicist best known for his work on planetary motion.

"The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics."

Ms. Sharp, with her comment, seems to imply that there is nothing foundational, there is nothing which is to be taken as true, and that order and harmony – if such things exist – must therefore be relative.

Unfortunately, many would not argue too strenuously with this notion. People often see mathematics as a compilation of miscellaneous facts and algorithms that must be memorized to be mastered. It is a subject where students are told order and harmony exist, but these seem murky at best. This failing has been present in the American educational system going back many years.

We must reject Ms. Sharp's implication that there is no foundation – no truth – in mathematics. In fact, mathematics builds on previous knowledge as no other field does. Concerning the structure of mathematics, Ron Aharoni in *Arithmetic for Parents* writes,

"There are other fields in which knowledge is built on previous knowledge, but in no other field do the towers reach such heights, nor do the topmost layers rely so clearly on the bottom ones. … The secret to proper teaching of mathematics lies in recognizing these layers and establishing them systematically."

There is no question that the mastery of basic skills is a major part of arithmetic. If elementary math facts are not memorized, each time a student moves to a higher concept too much time and energy will be wasted recalculating what could have been easily mastered. Additionally, memorization is a key component of the curriculum at just two points in elementary arithmetic. In first grade, students must master addition facts. In third grade, students must master multiplication facts. However, memorization is not the only part of arithmetic and the student must also have an understanding of how layers fit together, how the new concept follows from that which has been established.

When teaching Differential Calculus, very early in the semester I give what is called an *Algebra Mastery Exam*. Why is this valuable? To illustrate the concepts at the heart of calculus, those concepts need to be placed in an algebraic setting. At that point, students cannot afford difficulties with algebra. Algebra has become the tool used to open up the ideas which lead to the study of movement and rates of change. If students are still struggling with algebra and other necessary background material, they will be unable to appreciate or understand the new concepts being presented.

In the same way, when a student studies algebra, arithmetic becomes the tool to understanding. Without a solid foundation of arithmetic, the student will struggle in the setting of algebra and will therefore not fully understand the new concepts under consideration or how these new concepts follow from what should have been established.

Ron Aharoni continues his discussion of the structure of arithmetic,

"The same (structure) is true in elementary mathematics. However, since it deals with the bottom of the tower, the number of layers it establishes is smaller. There is no long chain of arguments as in higher mathematics. This is one of the reasons it is appropriate for children. Elementary school mathematics is not sophisticated, but it contains wisdom. It is not complex, but it is profound."

The profound nature and wisdom in arithmetic can be found at every stage of its study. It is seen in a study of the decimal system and its relationship with arithmetic operations. It can be seen in the time-tested algorithms for multi-digit addition, subtraction, and multiplication. There is wisdom in understanding how the long division algorithm is of a different nature of the other arithmetic algorithms. Finally, it is seen in the study of fractions, or parts of whole numbers.

As an illustration of the heights reached in arithmetic, let us consider division by fractions. The knowledge needed for understanding this concept is substantial. Required is a strong grasp of the meaning of fractions, along with the multiplication of fractions and inverse operations. Also needed for the study is an understanding of the meaning of division by whole numbers. These in turn require knowledge of multiplication of whole numbers leading down a path which includes mastery of addition and decimal system. To fully understand this sixth grade concept, one must follow chain of reasoning which ends all the way back in first grade mathematics. Reaching these heights in elementary school will provide students the solid background and understanding necessary for the study of upper school mathematics.

To fully understand division by fractions, one must go much deeper than the old saying,

"Ours is not to reason why, just invert and multiply."

This saying is still in use today; at least it is in use when the curriculum is rigorous enough to require computational proficiency of division by fractions. In *Knowing and Teaching Elementary Mathematics*, Liping Ma points to the lack of understanding in elementary school. From her study, only 43% of sampled teachers in the United States showed computational proficiency of a division by fractions problem and only 4% could display conceptual understanding by giving a word problem or model describing a division by fractions problem.

As a comparison, all of the Chinese teachers in Dr. Ma's study could display computational proficiency and 90% could display conceptual understanding. This should give us a strong indication of a fundamental reason China consistently outscores the United States on international exams in mathematics.

More importantly, if we are to lead our students to discover *'the rational order and harmony which has been imposed on (the natural world) by God'* by finding the same order and harmony in mathematics, and if we are to help our students *'see the nature of numbers with the mind only'*, then we must first have a deep understanding of the fundamental concepts of the subject and how those concepts fit together. How can we teach our students what we ourselves do not fully understand?

As educators, it is vital that we continue to strive to better understand, both procedurally and conceptually, the mathematics we teach. I have studied mathematics my entire life and, upon returning to its study, I am still amazed at the subtleties that I find in the relationships between the ideas of arithmetic. Developing a strong procedural and conceptual understanding is crucial to developing that understanding in a student.

It is not just in the successful continuation of the study of mathematics that we should consider our understanding of mathematics. The notion of foundational ideas – of truth – in education can probably be seen in the study of mathematics more clearly than in any other subject taught in many schools. If the foundation – the truth – is left out of mathematics, there will be few places for it to be seen in education at all.

In *Mathematics for the Non-mathematician*, Morris Kline writes of the importance of mathematics in preparing the mind for higher thought.

"The abstractions of mathematics possessed a special importance for the Greeks. The philosophers pointed out that, to pass from a knowledge of the world of matter to the world of ideas, man must train his mind to grasp the ideas. These highest realities blind the person who is not prepared to contemplate them. He is, to use Plato's famous simile, like one who lives continuously in the deep shadows of a cave and is suddenly brought out into the sunlight. The study of mathematics helps make the transition from darkness to light. Mathematics is in fact ideally suited to prepare the mind for higher forms of thought because on one hand it pertains to the world of visible things and on the other hand it deals with abstract concepts. Hence through the study of mathematics man learns to pass from concrete figures to abstract forms; moreover,

this study purifies the mind by drawing it away from the contemplation of the sensible and perishable and leading it to the eternal ideas."

By preparing the student's mind, by showing him the truth portrayed in mathematics, even in arithmetic, he will be ready not only for the study of higher level mathematics, but will also be better prepared to grasp truth in all areas of study.

Works Cited

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