The Purpose of Mathematics in a Classical Education

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A resurgence of interest in classical education has been evident in recent years. This has been due, in part, to a number of influential writings on regaining a "lost" knowledge in our culture which have, in turn, inspired an increasing number of schools and consortiums (such as those you represent) founded on a classical model.

When I first began surveying the landscape of classical education, it became evident that there is a reasonably clear vision available for the purpose of the study of humanities. Here, I am thinking of languages and great books. In history, the study of western civilization and the founding of our country. Not just what should be studied, but why the pursuit is vital to an education.

What did not seem as clear, though, is the nature the study of mathematics in this type of education. As a school sets forth on its mission of providing a classical education, what should be the aim of the study of mathematics?

How is mathematics to be approached? Is mathematics a science? Is it a set of skills to be memorized? Can the study of mathematics be more deeply integrated into a classical education? If so, is this necessary? Nearly everyone would agree that the study of mathematics belongs in a classical education, but the purpose of this study is not always clear.

Mathematics can be looked at from many different directions. I quote Morris Kline from his book *Mathematics for the Nonmathematician*:

"Perhaps we can see more easily why one should study mathematics if we take a moment to consider what mathematics is. Unfortunately the answer cannot be given in a single sentence or a single chapter. The subject has many facets or, some might say, is Hydra-headed. One can look at mathematics as a language, as a particular kind of logical structure, as a body of knowledge about number and space, as a series of methods for deriving conclusions, as the essence of our knowledge of the physical world, or merely as an amusing intellectual activity."

The view we take of mathematics affects the way in which we teach it and how it fits into the overall curriculum.

So how should the study of mathematics be viewed?

I believe Johannes Kepler, seventeenth century German physicist best known for his work on planetary motion, gave us a starting point when he wrote:

"The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics.… Just as the eye was made to see color and the ear to hear sounds, so the human mind was made to understand quantity."

One of the chief aims of mathematics has always been to reveal and describe an order in the natural world.

If we look back to the early days of mathematics, say four to five thousand years ago, we will see what the Egyptian and Babylonian civilizations offered in mathematics. They offered a very practical approach to mathematics answering questions that rarely extended beyond what was necessary to operate in daily life. During this time, rudimentary arithmetic and algebra were built up to answer questions in commerce and agriculture. The useful purposes for which they employed mathematics dealt with: Money exchange, simple and compound interest, computing wages, expressing weights and lengths, dividing inheritances, and determining volumes of granaries and areas of fields. Their mathematics was also used to study astronomy, making it possible to create calendars to accurately predict natural occurrences such as floods, something necessary for agricultural purposes. Accurate calendars could also be used for purposes of religious ceremony, such as, building temples so that the sun would shine on the altar at the appropriate day and time. These civilizations developed an elementary arithmetic, notation, some early algebra, and basic empirical formulas in geometry.

When considering classical mathematics, the Greeks must be a main focus of our attention. In fact, the primary reason for discussing earlier mathematics is to understand what the Greeks inherited and what they left to their posterity.

Whereas the Egyptians and Babylonians produced a fairly crude and very practical mathematics based on experience, the Greeks removed mathematics from its practical underpinnings. A major step in its advancement was the recognition that mathematics --in numbers and geometric figures -- can be dealt with in the abstract. This was not a small step in human thinking, and this initial step has been attributed to the Pythagorean School of ancient Greece.

Set up in the abstract, reasoning can now take over to form the basis of mathematical thought. One of the greatest gifts given to us by the Greeks is this gift of improved reasoning and this reasoning was brought into, honed, and perfected in mathematics. Mathematical results became a chain of propositions, arising out of preliminary assumptions, and advanced by reasoning. Probably the best illustration of this is Euclid's *Elements*, a set of thirteen volumes covering geometry of the plane and space as well as many results in number theory.

In approximately 300 B.C., Euclid brought together much of what was known in mathematics up to that point and organized it in such a way that, beginning with short list of abstract statements assumed to be true and armed with reasoning, he pieced this body of knowledge together as an extended chain. He did so in such a way that the *Elements* became the standard textbook in geometry for the next twenty-two hundred years. It is only in recent years -- approximately the last one hundred years -- that the *Elements* has been discarded as required reading for all educated people.

With the strengthening of the connection between mathematics and reasoning by the Greeks, mathematics would next become closely tied to philosophy, theology, and the natural sciences. Consider the following from Plato's *Republic* (Book VII):

"The knowledge at which geometry aims is knowledge of the eternal, and not of anything perishing and transient. Geometry will draw the soul towards truth, and create the spirit of philosophy, and raise up that which is now unhappily allowed to fall down. Therefore, nothing should be more sternly laid down than that the inhabitants of your fair city should by all means learn geometry."

As a country, we are not heeding this warning as far too few schools teach geometry in the spirit of Euclid.

Also from the *Republic* (Book VII) concerning the formation of leaders:

"We must endeavor that those who are to be the principal men of our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only; … arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible and tangible objects into the argument."

The abstraction that Greeks brought to mathematics should never be discounted in importance. I take this extended quote from Morris Kline, again from his book *Mathematics for the Nonmathematician:*

"The abstractions of mathematics possessed a special importance for the Greeks. The philosophers pointed out that, to pass from a knowledge of the <u>world of matter</u> to the <u>world of ideas</u>, man must train his mind to grasp the ideas. These highest realities blind the person who is not prepared to contemplate them. He is, to use Plato's famous simile, like one who lives continuously in the deep shadows of a cave and is suddenly brought out into the sunlight. The study of mathematics helps make the transition from darkness to light. Mathematics is in fact ideally suited to prepare the mind for higher forms of thought because on one hand it pertains to the world of visible things and on the other hand it deals with abstract concepts. Hence through the study of mathematics man learns to pass from concrete figures to abstract forms; moreover, this study purifies the mind by drawing it away from the contemplation of the sensible and perishable and leading it to the eternal ideas."

While it can be argued that mathematics as its own subject began with the Greeks, it certainly did not end there. After the Greek contributions, there was very little advancement in mathematics until the fifteenth century. In fact, until that time much of Greek mathematics had been lost to the Western world and it was during the Renaissance that much of it began to be recovered.

In the sixteenth and seventeenth century, figures such as Rene Descartes and Pierre de Fermat, pushed further by the desire to discover order in the universe, brought geometry out of abstraction and pure reasoning by introducing a new setting, what is known today as the

Cartesian coordinate system. Coordinate systems allowed more readily for measurements of natural events in the universe and paved the way for such greats as Newton and Leibniz who further developed these ideas in Calculus.

It was this period that helped usher in the “mathematization” of science, an ever-increasing attempt to discover order in the physical universe through the language of mathematics. This led to a tremendous growth in knowledge in both mathematics and science, but these advancements have come at a cost.

In modern times, mathematics is often seen as a language useful primarily to scientists, not just in the natural sciences, but increasingly in the social sciences. The honing of reasoning skills -- a critical component of a liberal education -- is often downplayed or completely neglected in a mathematics education. The cohesive structure of mathematics is often discarded to more quickly obtain the ‘useful’ results, but then these ‘useful’ results are often useless because this language of mathematics is without its cohesive structure and, therefore, the order of the universe remains unclear.

With the repeated emphasis on usefulness, it may be that modern mathematics education is much closer in the spirit to the Egyptians than the Greeks.

If you doubt this connection, you only need to look at the discussions concerning mathematics that take place in the public arena. Every argument for the purpose of its study seems to stem idea that it will help us in the ‘21st century economy”.

But that is not the true aim of the study of mathematics. Mathematics in a classical education will seek to promote the understanding of order and harmony in the universe. Mathematics, as a language, reveals this order and harmony, yet it should also be lifted from this concrete foundation and brought into the world of the abstract. The study of mathematics trains students in the context in which the discovery of its concepts arose as well as the reasoning which provides its structure. Although the study of mathematics has fallen well short this purpose in modern times, its implementation will strengthen an education, in particular a classical education.

Arithmetic: Leading the Mind toward Truth

If this is the aim of the study of mathematics in a classical education, I would next like to transition to the question of how we should approach the study of elementary school mathematics. As we will see, this is a critical part of the discussion.

It seems most people would prefer keep their attention on more advanced mathematics. For example, the number of calls to eliminate the study of algebra and geometry from the mathematics curriculum in favor of more 'applied subjects' has been on the rise in recent years (two such articles have appeared in the New York Times the past year-and-a-half). The thrust of the argument usual goes something like this, "If you look around, you will see students struggling with algebra and geometry. We have tried some things with these courses, but students are still struggling. It's time to get rid of them for the general public." There rarely seems to be a call to look closely at the elementary mathematics education which precedes these subjects. It may be that the real problem begins in the study of arithmetic, but, too often, we attempt to treat the symptoms instead of the core problem.

In the study of mathematics, the elementary school curriculum is of vital importance as it is necessary to first lay the proper foundation for further study. If the purpose of mathematics is to describe order in the universe and to hone reasoning skills, then elementary mathematics must prepare the student for such studies. Just as the typical first grader is not ready to read Shakespeare, the typical first grader is not ready to tackle the first book of Euclid's *Elements* or the quadratic formula.

To develop the critical thinking and problem solving skills necessary for the study of 'advanced' subjects, basic skills must be mastered and the child's mind prepared for this endeavor to be successful. In mathematics, this begins in the elementary grades with the study of arithmetic.

Arithmetic is the study of quantity and the relationships between quantities beginning with the decimal system for representing quantities (in particular whole numbers), progressing through the elementary operations of addition, subtraction, multiplication, and division, and extending into the study of fractions.

Arithmetic, at its most basic level, is an abstraction of the physical world. In some ways, it is one of the first formal abstractions to which a child is exposed, and its study is expected of every student. The meaning of a quantity and how it is represented is linked to reality. So is the meaning of an arithmetic operation. Certain situations lead to the addition of numbers, while others lead to subtraction. Still others, multiplication or division?

There are three meanings which can be attributed to an addition problem. Do we know them? They are increasing, combining, and adding the difference. If we do not know them and the subtle difference each meaning represents, we will not understand the complete picture of addition or why students almost universally struggle with one in particular, adding the difference.

To go further, each of these meanings corresponds to meaning in subtraction. Beginning to understanding analogues and differences at this level will help students as they progress in the subject.

It is often forgotten that there is not one, but two concrete meanings describing division (they are 'sharing division' and 'containment division'). It is a lack of the second concrete model for division that leads to many of the difficulties in assigning meaning when the time comes to divide by fractions. Without this understanding we are left to rely solely on rote memorization of an algorithm for our knowledge in division by fractions. We will return to this example toward the end of the talk.

This connection between the concrete and the abstract is one of the aspects of mathematics that had the ancient Greeks singing its praises. Mathematics is grounded on the concrete level, and this grounding is vital to initial understanding. But, the concrete can be discarded and knowledge advanced through reasoning. We should not forget, though, that we can (and occasionally must) return from the abstract to the concrete to 'test' our reasoning and understanding. This is how we prepare children's minds for higher thought. This connection between the concrete and the abstract displays an undeniable truth and beauty in mathematics for those who seek it.

How is the study of arithmetic to be approached? Consider the following quote from the 1960s book *A Parent's Guide to the New Math* by Evilyn Sharp:

"If you read in your child's book that 3+4=2, don't be startled. It does in the modulo five system. Also 4+1=0, 3+3=1, and 4+4=3. In other words, nothing is sacred anymore."

"Nothing is sacred anymore."

This can be contrasted with the words of Johannes Kepler which we heard earlier:

"The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics."

Ms. Sharp, with her comment, seems to imply that there is not really a foundation, there is nothing which is true, and that order and harmony – if such things even exist – must therefore be relative.

Unfortunately, many Americans would not argue too strenuously with this notion. People often see mathematics as a compilation of miscellaneous facts and algorithms that must be memorized to be mastered. It is a subject where students are told order and harmony exist, but these seem murky at best. This failing has been present in the American educational system going back many years. While student's difficulties become truly evident during the study of algebra, they begin at a much earlier stage.

I believe we must reject Ms. Sharp's implication that there is no foundation -- no truth -- in mathematics. In fact, mathematics builds on previous knowledge as no other field does. Concerning the structure of mathematics, Dr. Ron Aharoni (an Israeli mathematician) in his book *Arithmetic for Parents* states:

"There are other fields in which knowledge is built on previous knowledge, but in no other field do the towers reach such heights, nor do the topmost layers rely so clearly on the bottom ones. … The secret to proper teaching of mathematics lies in recognizing these layers and establishing them systematically."

There is no question that the mastery of basic skills is a major part of arithmetic. There is no way around memorization. If elementary math facts are not memorized, every time a student moves to a higher concept too much time and energy will be wasted recalculating what could have been easily mastered. Additionally, memorization is a key component of the curriculum at just two points in elementary arithmetic. In first grade, students must master basic addition facts. In third grade, students must master basic multiplication facts.

But, memorization is not the only part of arithmetic and the student must also have an understanding of how layers fit together, how the new concept follows from that which has been established.

When teaching first semester Calculus, very early in the semester I give what is called an *Algebra Mastery Exam*. Why? Because at that point, to study the ideas and concepts which are at the heart of what we call Differential Calculus, the basic concepts need to be placed in an algebraic setting. At that point, students cannot afford difficulties with algebra. Algebra has become the tool used to open up the ideas which lead to the study of movement and rates of change. If students are still struggling with algebra and other necessary background material, they will be unable to appreciate or understand the new concepts being presented.

In the same way, when a student studies algebra, elementary arithmetic becomes the tool to understanding. Arithmetic must have been previously mastered. Without a solid foundation of arithmetic, the student will struggle in the setting of algebra and will therefore not fully understand the new concepts under consideration or how these new concepts follow from what should have been established.

Dr. Aharoni continues his discussion of the structure of arithmetic:

"The same (structure) is true of elementary mathematics. However, since it deals with the bottom of the tower, the number of layers it establishes is smaller. There is no long chain of arguments as in higher mathematics. This is one of the reasons it is appropriate for children. Elementary school mathematics is not sophisticated, but it contains wisdom. It is not complex, but it is profound."

The profound nature and wisdom in arithmetic can be seen at every stage of its study. Revealing and examining it is one of the main focuses of the seminar 'The Teaching of Elementary Mathematics' which was referred to by Emilie in her introduction. I would like to give a brief survey of examples which illustrates this profound nature.

The decimal system is regularly used in daily life. What is the meaning of twenty-three? At its foundation, our number system groups objects in tens. Why? Is it because it is natural to do so? No, but it does provide an immediate order and efficiency which provides a necessary foundation as the study of arithmetic progresses.

Without the decimal system, I cannot easily work with large numbers.

By grouping into tens and having a number system which allows this representation, when I discuss twenty-three students, everyone knows I mean two groupings of ten and three additional students. I can discuss the population of the United States using only 9 digits. I can discuss the number of stars in our galaxy (400 billion and incredible number) using only 12 digits. We can discuss our national debt using only 14 digits. These groupings may not be natural, but it is incredibly efficient.

This is potentially the idea to which Ms. Sharp (of the New Math quote) was trying to refer. Grouping into tens is actually somewhat arbitrary. This is only a convention and not a fundamental concept of arithmetic. Groupings of fives could have been used instead. Babylonians grouped in sixties instead of tens and the Greeks inherited these groupings of sixty. But, groupings of tens are used in the modern world and once a student truly understands arithmetic in this context (called base 10), they can move on to understanding groupings of other bases. When time is spent understanding what it means to work in a new base (or new set of groupings) it can be seen that changes are primarily superficial in nature. And nothing 'sacred' is given up.

Without mastery in groupings of tens, the study of fives or sixties will seem just as mysterious as tens and no new insight will be gained from the study.

When a student moves on to the two-digit addition algorithm with regrouping (so-called column addition), a complete understanding of the decimal system is necessary. When the arithmetic problem leads to too many ones, a 'regrouping' into a group of tens is needed (i.e., 28+33=61). If the student never understands place values in the decimal system, then he will struggle with this concept. Also, if one-digit math facts have not been mastered, then a significant amount of time and energy will be spent on basic math facts instead of on the new concept of 'regrouping'.

The continued study of arithmetic will lead the student to the algorithm for two-digit multiplication.

I would like to pause here for a moment and consider the following question: 'Is knowing an algorithm (a procedure for calculation) the same as understanding the concepts behind the algorithm'? No, it is not, but each helps to provide a deeper understanding of the other. It is important to teach algorithms and it is just as important to teach students the reason why the algorithm is true.

To succeed in the study of the algorithm for two-digit multiplication, a mastery of one-digit multiplication and multi-digit addition is needed along with a solid understanding of the decimal system and concept of regrouping. If any of these pieces are missing, the algorithm cannot make sense. Additional concepts of multiplication by tens and the distributive property are also necessary to fully understand two-digit multiplication. Without a complete understanding, this will become one more algorithm to be memorized and its relationship with the rest of the body of knowledge of arithmetic will remain unclear.

This structure of concepts goes together in what math educator Dr. Liping Ma in her book *Knowing and Teaching Elementary Mathematics* calls 'knowledge packets'. Each piece fits with every other to help tie together the cohesive body of knowledge. If anything is missing, the student will struggle with the new concepts until the gaps are filled in and the foundation is made solid.

Mastery must be attained at each level or the subject ultimately becomes an unrelated mess of memorization. As educators, this is certainly something we want to avoid. With the appropriate material mastered, though, students can understand why the multi-digit multiplication algorithm works and how it relates to the whole of mathematical knowledge seen in the previous four years.

In her study from the mid 1990s, Dr. Ma goes on to show that while her entire sample of elementary teachers in the United States correctly knew the algorithm for multi-digit multiplication, only 40% had a conceptual understanding of how and why the algorithm worked. This was compared with conceptual understanding by 92% of the Chinese teachers in the study. It is not a coincidence that Chinese students have been consistently scoring much better than students from the United States on international exams. Also, it needs to be pointed out that while teachers in the United States typically receive 16 to 18 years of formal education; Chinese teachers typically receive 11 to 12 years of schooling (through ninth grade and another 2 to 3 years in normal school).

Let us return for a moment to Dr. Aharoni's earlier quote:

"Elementary school mathematics is not sophisticated, but it contains wisdom. It is not complex, but it is profound."

It is vital that teachers continue to strive to better understand, both procedurally and conceptually, the mathematics they will teach. I have studied mathematics my entire life and, upon returning to this study, I am still amazed at the subtleties that I can find in the relationships between concepts in arithmetic. Developing a strong procedural and conceptual understanding is crucial to developing understanding in a student.

Holding this thought aside for a few moments, I would like to shift gears to see what lessons can be learned by taking a brief look at the history of math education in the United States since the beginning of the twentieth century. For anyone interested, a more thorough look can be found in the article "A Brief History of American K-12 Mathematics Education in the 20th Century" by mathematician David Klein (available: http://www.csun.edu/~vcmth00m/AHistory.html).

Early in the twentieth century, the Progressive movement began to exert enormous influence in mathematics education in the United States through education professor William Kilpatrick, a protégé of John Dewey. One of the results of this influence was the dumbing down of mathematics education by rejecting the notion that the study of mathematics contributed to mental discipline and subscribing to the idea that mathematics should only be taught to students based on direct practical value. Kilpatrick proposed that the study of algebra and geometry in high school be discontinued "*except as an intellectual luxury*." As a result, the percent of high school enrollment in traditional math courses plummeted during the first half of the century. In algebra, enrollment went from 57% in 1910 to 25% in 1955. Over that same time period, geometry enrollments went from 31% to 11%.

Kilpatrick saw mathematics as *"harmful rather than helpful to the kind of thinking necessary for ordinary living"*. The deterioration of mathematics education at all levels in K-12 continued during this time and could be seen clearly in incidents such as one in the forties when army recruits knew so little math that the army had to provide training in the arithmetic needed for basic bookkeeping and gunnery.

If you believe that Kilpatrick's comments are outmoded, I would like to draw your attention to an Op-Ed piece that appeared in the New York Times this past fall. In the article 'Is Algebra Necessary?', Andrew Hacker, professor emeritus of political science at Queens College, City University of New York, writes:

'A TYPICAL American school day finds some six million high school students and two million college freshmen struggling with algebra. In both high school and college, all too many students are expected to fail. Why do we subject American students to this ordeal? I've found myself moving toward the strong view that we shouldn't.'

He goes on to blame the subject of Algebra (specifically) for many of the ills in public education (including drop-out rates) He suggests an alternative course of study, but this course of study cannot really be accessed without a background in algebra.

The closing lines of the article state: "It would be far better to reduce, not expand, the mathematics we ask young people to imbibe." This is an echo of Kilpatrick's comments from nearly one hundred years ago, and its sentiment seems to be regaining a more widespread acceptance.

In the fifties and sixties, the 'New Math' arose. The movement was in response to several factors, including the progressive dumbing down of the mathematics curriculum during the first half of the century and a national wake-up call due to the launch of Sputnik by the Soviet Union.

'New Math' had flaws of its own. From Albert Meder, Jr, the executive director of the commission which implemented 'New Math' we get the following quote:

"If you dug up an old seventeenth-century don he could walk right into any classroom and start teaching math. Nobody would notice the difference, since the content of the courses hadn't changed in the last three hundred years."

He went on to say that mathematics was the only subject for which this was true. It could not happen in history, language, or science. (from *A Parent's Guide to the New Math* by Evilyn Sharp)

Mr. Meder and the 'New Math' commission failed to understand that mathematics is not like other fields. Euclid's *Elements* was not used as the primary textbook in geometry for twenty-two centuries because all who came after him lacked the intelligence and creativity to explain the subject or dig deeper for further results. Euclid did such an extraordinary job compiling knowledge and laying out the subject in such a systematic way that mathematicians did not feel the need to tamper with it until the late nineteenth century brought a fundamental shift in philosophy. He laid a solid foundation and built up a strong body of knowledge.

The fundamental problem with the 'New Math' curriculum of the sixties was not in the 'dumbing down' of mathematics education, but in its failure to lay a solid foundation. The foundation was hastily constructed using concepts too abstract for most children to understand. This was done in an attempt to reach greater mathematical heights as quickly as possible. The result was a shaky foundation that most children were unable to build upon. 'New Math' was quickly abandoned, but there lessons learned from this episode should not be quickly forgotten.

Since that time, educational systems in the United States have seen a renewal in progressive influence. I am sure this is not news to you. The claim by progressive educators is they desire to teach students 'how to think' instead of 'what to think'. Such educators embrace 'hands-on' and 'experiential' opportunities in the classroom where students can 'construct their own mathematics' and draw their own conclusions. Existing mathematical knowledge is deemed unimportant – or even potentially harmful – in the process.

As I have tried to emphasis, in the study of mathematics, existing knowledge is a vital component of the curriculum and must be built up systematically. The student must master basic skills with speed and proficiency to lay a solid foundation for further study. By understanding the structure of mathematics at this level, the student is prepared to begin tying together these strands of mathematical knowledge as they move toward a more conceptual understanding of mathematics. Without a basis of knowledge there is nothing to build upon.

It seems that modern progressive education practices promote the idea that elementary mathematics is not profound, it does not contain wisdom.

But, arithmetic is not simple, just as the trivium is not trivial.

Let me offer a final illustration of the profound depth available in arithmetic, if one only seeks it. The most difficult concept in elementary arithmetic is division by fractions, typically introduced in the fifth grade. The knowledge needed for understanding this concept is substantial. Required is a thorough understanding of fractions, the multiplication of fractions, and the division by whole numbers. These in turn require an understanding of multiplication of whole numbers leading down the path taken earlier which includes a thorough understanding of addition and decimal system. A chain of reasoning can be followed which ends all the way back in first grade mathematics. Reaching these depths in elementary school will provide students the solid background and understanding necessary for the study of upper mathematics.

Conceptual understanding of division by fractions is much deeper than the old saying:

"Ours is not to reason why, just invert and multiply."

This saying is still in use today; at least it is in use when the curriculum is rigorous enough to require computational proficiency of division by fractions. Dr. Liping Ma's study in *Knowing and Teaching Elementary Mathematics* points to the lack of conceptual understanding in elementary school. From the study, only 43% of sampled US teachers had computational proficiency of a division by fraction problem and only 4% could display the conceptual understanding by giving a word problem describing the meaning of the division by a fraction problem.

As a comparison, all of the Chinese teachers in Dr. Ma's study could display computational proficiency and 90% could display conceptual understanding by creating a story problem related to the division by fractions problem. This is indicative of why China consistently outscores the US on international exams.

I believe Dr. Ma's book should be part of any program preparing teachers for the teaching of elementary mathematics. It is required reading in my seminar. In it, she offers a look into how each group approaches and understands the material they are teaching. In our society, it has all too commonplace to think of arithmetic as trivial. Try telling someone that they have something new to learn about addition and that their understanding is not deep enough.

Do not look for leadership in this area from the math education community. In fact, a survey of math education programs around the country showed that while many departments use Dr. Ma's research in course preparation, most did not use it as assigned reading within courses for fear that the students (in this case, prospective teachers) would 'have difficulty with the math content' and 'exposing them to this content would make prospective teachers too uncomfortable with the fact that they do not know too much math.'

No amount of work in pedagogy can make up for the lack of understanding of a subject.

Using a content-rich curriculum (and not all curriculum are created equal) and investing in the professional development of teachers, in particular in the area of subject knowledge, has been shown to overcome many of these difficulties and provides a more solid basis for student learning and their continued study of advanced mathematics.

But, it is not just in the continuing study of mathematics that we should address these issues. The concept of foundational ideas – of Truth in education – can probably be seen in the study of mathematics more clearly than in any other subject taught in many of our schools, this is especially true of public school. If the foundation – the truth – is taken out of mathematics, it will inevitably be the case that it is seen nowhere in a K-12 education. By preparing the student's mind, by showing him the truth portrayed in mathematics, he will be ready not only for the study of higher level mathematics, but will also be prepared to look for this in other areas of study.

Thank you for your time and I would be happy to entertain questions or comments that you might have.